

A Limit Theorem for Products of Free Unitary Operators

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Abstract

This paper establishes necessary and sufficient conditions for the products of freely independent unitary operators to converge in distribution to the uniform law on the unit circle.

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1 Introduction

Suppose X is a unitary n -by- n matrix. Then X has n eigenvalues, which are all located on the unit circle. If we give each eigenvalue a weight of n^{-1} , then we can think about the distribution of these eigenvalues as a probability distribution supported on n points of the unit circle. More generally, if X is a unitary operator in a finite von Neumann algebra, then we can define a spectral probability distribution of X , which is supported on the unit circle (see, e.g., Section 1.1 in Hiai and Petz [2000]).

If we have several unitary operators X_1, \dots, X_n , then it is natural to ask about the spectral distribution of their product. In general, we cannot determine this distribution without more information about relations among operators X_1, \dots, X_n . However, if X_1, \dots, X_n are infinite-dimensional and, in a certain sense, in a general position relative to each other, then the spectral distribution of their product is computable. The idea of a general position was

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formalized by Voiculescu in his concept of freeness of operators (see Voiculescu [1983], Voiculescu [1986], and a textbook by Hiai and Petz [2000]). If operators X_1, \dots, X_n are free and unitary and their spectral probability distributions are μ_1, \dots, μ_n , respectively, then the distribution of their product is determined uniquely. This distribution is called the free multiplicative convolution of measures μ_1, \dots, μ_n and denoted as $\mu_1 \boxtimes \dots \boxtimes \mu_n$.

What can we say about the asymptotic behavior of $\mu^{(n)} =: \mu_1 \boxtimes \dots \boxtimes \mu_n$, as n increases to infinity? In particular, what are necessary and sufficient conditions on μ_i that ensure that $\mu^{(n)}$ converges to the uniform distribution on the unit circle?

To answer this question, let us define the expectation with respect to the measure μ_i . This is a functional that maps functions analytic in a neighborhood of the unit circle to complex numbers:

$$E_{\mu_i} f =: \int_{|\xi|=1} f(\xi) d\mu_i(\xi).$$

If unitary operator X_i has the spectral probability distribution μ_i , then we will also write:

$$Ef(X_i) =: E_{\mu_i} f.$$

In particular, EX_i denotes $\int_{|\xi|=1} \xi d\mu_i(\xi)$. Then the answer is given by the following theorem:

Theorem 1 *Suppose $\{X_i\}_{i=1}^\infty$ are free unitary operators with spectral measures μ_i . The measures $\mu^{(n)}$ of the products $\Pi_n =: X_1 \dots X_n$ converge to the uniform measure on the unit circle if and only if at least one of the following situations holds:*

- (i) *There exist two indices $i \neq j$ such that $EX_i = EX_j = 0$;*
- (ii) *There exists exactly one index i such that $EX_i = 0$, and $\prod_{k=i+1}^n EX_k \rightarrow 0$ as $n \rightarrow \infty$;*
- (iii) *There exists exactly one index i such that X_i has the uniform distribution;*
- (iv) *$EX_k \neq 0$ for all k , and $\prod_{k=1}^n EX_k \rightarrow 0$ as $n \rightarrow \infty$.*

In other words, convergence of $\mu^{(n)}$ to the uniform law implies that $\prod_{k=1}^n EX_k \rightarrow 0$, and the only case when the reverse implication fails is when $EX_i = 0$ for exactly one X_i , the measure μ_i is not uniform, and $\prod_{k=i+1}^n EX_k \not\rightarrow 0$ as $n \rightarrow \infty$. Note that cases (ii) and (iii) above are not exclusive. It may happen that both μ_i is uniform and $\prod_{k=i+1}^n EX_k \rightarrow 0$ as $n \rightarrow \infty$. In this case, both (ii) and (iii) hold, and $\mu^{(n)}$ converges to the uniform law.

This theorem can be thought of as a limit theorem about free multiplicative convolutions of measures on the unit circle. There is some literature about traditional multiplicative convolutions of measures on the unit circle, or more generally, about convolutions of measures on compact groups. For the unit

circle, this investigation was started by Levy [1939]. Then it was continued by Kawada and Itô [1940], who studied compact groups, and Dvoretzky and Wolfowitz [1951] and Vorobev [1954], who both considered the case of commutative finite groups. These researchers found an important necessary condition for convergence of convolutions to the uniform law. This condition requires that there should be no normal subgroup such that the convolved measures are supported entirely in an equivalence class relative to this subgroup. This condition is sufficient if summands are identically distributed. If they are not, then there are some sufficient and necessary conditions, which are especially useful if the group is cyclic. A textbook presentation with further references can be found in Grenander [1963].

Recent investigations of convolutions on groups are mostly concerned with the speed of convergence of convolved measures to the uniform law. For a description of progress in this direction, the reader can consult surveys in Diaconis [1988] and Saloff-Coste [2004].

It turns out that free convolutions converge to the uniform law under much weaker conditions than usual convolutions. As an example, consider the distributions that are concentrated on -1 and $+1$. Let measure μ_k put the weight p_k on $+1$. Then usual convolutions remain concentrated on -1 and $+1$, and therefore they have no chance to converge to the uniform distribution on the unit circle. In contrast, we will show that free convolutions do converge to the uniform law, provided that either $\prod_{k=k_0}^n (2p_k - 1) \rightarrow 0$ for arbitrarily large k_0 , or there exist two indices i and j such that $p_i = p_j = 1/2$.

The rest of the paper is organized as follows. Section 2 provides the necessary background. In Section 3 we outline the proof. Section 4 derives some auxiliary results that will be used in the proof. Section 5 proves the main result (Theorem 1). Section 6 derives the key estimate used in the proof. And Section 7 concludes.

2 Definitions and Background

Definition 2 *A non-commutative probability space is a pair (\mathcal{A}, E) , where \mathcal{A} is a unital C^* -algebra of bounded linear operators acting on a complex separable Hilbert space and E is a linear functional from \mathcal{A} to complex numbers. The operators from algebra \mathcal{A} are called non-commutative random variables, or simply random variables, and the functional E is called the expectation.*

The linear functional E is assumed to satisfy the following properties (in addition to linearity): i) $E(I) = 1$; ii) $E(A^*) = \overline{E(A)}$; iii) $E(AA^*) \geq 0$; iv) $E(AA^*) = 0$ implies $A = 0$; and v) if $A_n \rightarrow A$ then $E(A_n) \rightarrow E(A)$, where convergence of operators is in norm.

If $P(d\lambda)$ is the spectral resolution associated with a unitary operator A , then we can define a measure $\mu(d\lambda) = E(P(d\lambda))$. It is easy to check that μ is a probability measure supported on the unit circle. We call this measure, μ , the *spectral probability measure associated with operator A and expectation E* .

The most important concept in free probability theory is that of free independence of non-commuting random variables. Let a set of r.v. A_1, \dots, A_n be given. With each of them we can associate an algebra \mathcal{A}_i , which is generated by A_i ; that is, it is the closure of all polynomials in variables A_i and A_i^* . Let \bar{A}_i denote an arbitrary element of algebra \mathcal{A}_i .

Definition 3 *The algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ (and variables A_1, \dots, A_n that generate them) are said to be freely independent or free, if the following condition holds:*

$$\varphi(\bar{A}_{i(1)} \dots \bar{A}_{i(m)}) = 0,$$

provided that $\varphi(\bar{A}_{i(s)}) = 0$ and $i(s+1) \neq i(s)$.

For more information about non-commutative probability spaces and free operators we refer the reader to Sections 2.2 - 2.5 in the book by Voiculescu et al. [1992].

We will use two results regarding the free operators, which we cite without proofs. The first one is formula (2.2.3) on page 44 in Hiai and Petz [2000].

Proposition 4 *Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be free sub-algebras of \mathcal{A} , and let A_1, \dots, A_n be a sequence of random variables, $A_k \in \mathcal{A}_{i(k)}$ such that $i(k) \neq i(k+1)$. Then*

$$E(A_1 \dots A_n) = \sum_{r=1}^n \sum_{1 \leq k_1 < \dots < k_r \leq n} (-1)^{r-1} E(A_{k_1}) \dots E(A_{k_r}) E\left(A_1 \dots \hat{A}_{k_1} \dots \hat{A}_{k_r} \dots A_n\right), \quad (1)$$

where $\hat{}$ denotes terms that are omitted.

Remark: Note that on the right-hand side the expectations are taken of the products that have no more than $n-1$ terms. So a recursive application of this formula reduces computation of $E(X_1 \dots X_n)$ to a polynomial in the moments of the individual variables.

The second result is the Voiculescu multiplication theorem. To formulate it we need some additional definitions.

Define the ψ -function of a bounded random variable X as

$$\psi_X(z) =: \sum_{k=1}^{\infty} E(X^k) z^k. \quad (2)$$

If X is unitary operator with the spectral measure μ , then we can write:

$$\psi_\mu(z) = \int_{|\xi|=1} \frac{1}{1-\xi z} d\mu(\xi) - 1.$$

It is very useful to note that the ψ -function is related to the Poisson transform of the measure μ . Indeed, since

$$\operatorname{Re} \frac{1}{1-\xi z} = \frac{1-r \cos(\omega-\theta)}{1-2r \cos(\omega-\theta)+r^2},$$

where $\xi = e^{-i\theta}$ and $z = re^{i\omega}$, therefore,

$$\operatorname{Re} \frac{1}{1-\xi z} = \frac{1}{2} + \pi P(r, \omega - \theta)$$

where $P(r, \theta)$ is the *Poisson kernel*:

$$P(r, \theta) = \frac{1}{2\pi} \frac{1-r^2}{1-2r \cos \theta + r^2}.$$

Recall that the *Poisson transform* of a measure μ supported on the unit circle is defined as

$$U_\mu(z) =: \int_{-\pi}^{\pi} P(r, \omega - \theta) d\mu(\theta),$$

where $z = re^{i\omega}$. (Here we have identified measures on the unit circle and on the interval $[-\pi, \pi)$: $\mu(d\theta) = \mu\{\xi : |\xi| = 1 \text{ and } \arg \xi \in d\theta\}$). Hence,

$$U_\mu(z) = \frac{1}{\pi} \operatorname{Re} \psi(z) + \frac{1}{2\pi}. \quad (3)$$

Let $\psi_X^{-1}(u)$ denote the functional inverse of $\psi_X(z)$ in a neighborhood of $z = 0$, where $\psi_X(z)$ is as defined in (2). (This inversion is possible provided that $E(X) \neq 0$.) Define also

$$S_X(u) = \frac{u+1}{u} \psi_X^{-1}(u).$$

Theorem 5 [Voiculescu] *Suppose X and Y are bounded free random variables. Suppose also that $E(X) \neq 0$ and $E(Y) \neq 0$. Then*

$$S_{XY}(z) = S_X(z) S_Y(z).$$

The original proof can be found in Voiculescu [1987]. A simpler proof was given by Haagerup [1997]. Using this theorem, it is possible to compute the free convolution of two measures, μ_1 and μ_2 . First, we can compute their ψ -functions, ψ_{μ_1} and ψ_{μ_2} . Then we invert them and obtain the S -functions, S_{μ_1} and S_{μ_2} . Their product is the S -function of the free convolution, $S_{\mu_1 \boxplus \mu_2}$, and we can compute $\psi_{\mu_1 \boxplus \mu_2}$ by inversion. This determines the Poisson transform of $\mu_1 \boxplus \mu_2$, from which we can determine the measure itself. (For the one-to-one relation of Poisson transforms and corresponding measures, see Theorem I.3.1 on page 15 and a comment on page 20 in Garnett [1981].)

3 Outline of the Proof

Let Π_n denote the partial products: $\Pi_n = X_1 \dots X_n$. We denote $E(X_i)$ as a_i , and $E(\Pi_n)$ as $a_{(n)}$. First, note that it is enough to consider the case when all a_i are real and non-negative. Indeed, for an arbitrary sequence of real constants θ_n , the sequence of operators $e^{i\theta_n}\Pi_n$ converges in distribution to the uniform law if and only if the sequence Π_n converges in distribution to the uniform law. (Indeed, if, say, Π_n does not converge in distribution to the uniform law, then we can find an integer k such that $\left| \int_0^{2\pi} e^{ik\theta} \mu_{(n)}(d\theta) \right| \not\rightarrow 0$, where $\mu_{(n)}$ denotes the measure of Π_n . But then $\left| \int_0^{2\pi} e^{ik\theta} e^{i\theta_n} \mu_{(n)}(d\theta) \right| = \left| \int_0^{2\pi} e^{ik\theta} \mu_{(n)}(d\theta) \right| \not\rightarrow 0$, and this implies $e^{i\theta_n}\Pi_n$ does not converge in distribution to the uniform law.) Therefore if $a_i = E(X_i)$ is not real and positive, then we can replace X_i with $e^{-i \arg a_i} X_i$ without affecting the convergence of Π_n .

We divide the analysis into the following cases:

Case I $a_{(n)} \not\rightarrow 0$.

Case II $a_{(n)} \rightarrow 0$, and there are at least two indices, i and j , such that $a_i = a_j = 0$.

Case III $a_{(n)} \rightarrow 0$, and for all i , $a_i > 0$.

Subcase III.1 $\liminf a_i = 0$.

Subcase III.2 $\liminf a_i = \underline{a} > 0$.

Case IV $a_{(n)} \rightarrow 0$, and there exists exactly one index i , such that $a_i = 0$.

We will show that without loss of generality we can assume in this case that $a_1 = 0$, and $a_k > 0$ for all $k > 1$.

Subcase IV.1 X_1 has the uniform distribution.

Subcase IV.2 X_1 does not have the uniform distribution, and $\prod_{k=2}^n a_k \rightarrow 0$ as $n \rightarrow \infty$.

Subcase IV.3 X_1 does not have the uniform distribution, and $\prod_{k=2}^n a_k \not\rightarrow 0$ as $n \rightarrow \infty$.

We will show that Π_n does not converge to the uniform law if and only if either Case I or Case IV.3 holds.

4 Auxiliary Lemmas

We will need to perform functional inversions. A useful tool for doing this is Lagrange's formula.

Lemma 6 [Lagrange's inversion formula] Suppose that (i) f is a function of a complex variable z , which is analytic in a neighborhood of $z = 0$, (ii) $f(0) = 0$, and (iii) $f'(0) = a \neq 0$. Then the functional inverse of $f(z)$ is well defined in a neighborhood of 0, and the Taylor series of the inverse is given by the following

formula:

$$f^{-1}(u) = \frac{u}{a} + \sum_{k=2}^{\infty} \left[\frac{1}{k} \operatorname{res}_{z=0} \frac{1}{f(z)^k} \right] u^k,$$

where $\operatorname{res}_{z=0}$ denotes the Cauchy residue at 0. In addition,

$$f^{-1}(u) = \frac{u}{a} + \sum_{k=2}^{\infty} \left[\frac{1}{2\pi i k} \oint_{\gamma} \frac{dz}{f(z)^k} \right] u^k,$$

where γ is a circle around 0, inside which f has only one zero.

For a proof see Section 7.32 in Whittaker and Watson [1927].

We will also use the lemmas below:

Lemma 7 Suppose A and B are free unitary operators, $|E(A)| \leq a$ and $|E(B)| \leq b$. Then for all integer $k \geq 1$,

$$\left| E \left[(AB)^k \right] \right| \leq M_k \max(a, b)$$

for certain constants M_k , which depend only on k .

Proof: If we expand $E \left[(AB)^k \right]$ using Proposition 4, then we can observe that each term in the expansion contains either $E(A)$ or $E(B)$ as a separate multiple. The remaining multiples in this term are ≤ 1 in absolute value; therefore, we can bound each term by $\max(a, b)$. The number of terms in this expansion is bounded by a constant, M_k . Therefore, $\left| E \left[(AB)^k \right] \right|$ is bounded by $M_k \max(a, b)$. QED.

In the following lemmas we use the fact that the sequence of probability measures μ_i , supported on the unit circle, converges to the uniform law if and only if all their moments converge to 0, that is, iff for each $k \geq 1$, $\int_{|\xi|=1} \xi^k d\mu_i(\xi) \rightarrow 0$ as $i \rightarrow \infty$. For completeness we give a proof of this result.

Let us define $c_k^{(i)} =: \int_{|\xi|=1} \xi^k d\mu_i(\xi)$. Note that for a fixed i , $c_k^{(i)}$ are coefficients in the Taylor series of $\psi_i(z)$, i.e., the ψ -function of the measure μ_i .

Lemma 8 Let μ_i be a sequence of measures supported on the unit circle. If for each k the coefficients $c_k^{(i)} \rightarrow 0$ as $i \rightarrow \infty$, then $\psi_i(z) \rightarrow 0$ uniformly on compact subsets of the open unit disc.

Proof: Let Ω be a compact subset of the open unit disc, and let $\Omega \subset D_r$, where D_r denotes a closed disc with the radius $r < 1$. Fix an $\varepsilon \in (0, 1)$. Then we can find such a k_0 that

$$\left| \sum_{k=k_0}^{\infty} c_k^{(j)} z^k \right| < \varepsilon/2$$

for all $z \in D_r$ and all j . Indeed, $|c_k^{(j)}| \leq 1$, and therefore,

$$\left| \sum_{k=k_0}^{\infty} c_k^{(j)} z^k \right| \leq \frac{r^{k_0}}{1-r},$$

so we can take k_0 to be any integer greater than or equal to $\log(\varepsilon(1-r)/2)/\log r$.

Given k_0 , we choose a j_0 so large that for all $j > j_0$ and all $k < k_0$, we have $|c_k^{(j)}| < \varepsilon/(2k_0)$. This is possible because by assumption for each k coefficients $c_k^{(j)}$ converge to zero as $j \rightarrow \infty$, and we consider only a fixed finite number of possible k .

Consequently,

$$\left| \sum_{k=1}^{k_0-1} c_k^{(j)} z^k \right| \leq \sum_{k=1}^{k_0-1} |c_k^{(j)}| < \varepsilon/2$$

for every $j > j_0$ and all $z \in D_r$. Therefore,

$$\left| \sum_{k=1}^{\infty} c_k^{(j)} z^k \right| < \varepsilon$$

for every $j > j_0$ and all $z \in D_r$. Therefore, $\psi_j(z) \rightarrow 0$ uniformly on D_r , and therefore on Ω . Since Ω was arbitrary, we have proved that $\psi_j(z) \rightarrow 0$ uniformly on compact subsets of the unit disc. QED.

The fact that $\psi_j(z) \rightarrow 0$ implies that the Poisson transforms of measures μ_j converge to $\frac{1}{2\pi}$, and therefore $\mu_j \rightarrow \nu$, where ν is the uniform measure on the unit disc. Indeed, we only need to invoke the following result:

Proposition 9 *If Poisson transforms $U_{\mu_j}(z) \rightarrow 1/(2\pi)$ uniformly on compact subsets of the unit disc, then μ_j weakly converges to ν , where ν is the uniform probability measure on the unit circle.*

Proof: This proposition directly follows from Theorem I.3.1 on page 15 in Garnett [1981], adapted to the case of measures on the unit disc. QED.

Lemma 10 *Suppose $\{A_n\}_{n=1}^{\infty}$ is a sequence of unitary operators that converges in distribution to the uniform law. Let $\{B_n\}_{n=1}^{\infty}$ be another sequence of unitary operators, and let the operator B_n be free of the operator A_n for every n . Then the sequence of products $B_n A_n$ converges in distribution to the uniform law. Also, the sequence $A_n B_n$ converges to the uniform law.*

Proof: Let $a_k^{(n)} =: E\left((A_n)^k\right)$. By assumption, for each fixed k , the moment $a_k^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. If we represent $E\left((B_n A_n)^k\right)$ as a polynomial in individual moments of B_n and A_n , then all terms of this polynomial contain at least one of

the moments $a_i^{(n)}$, $i \leq k$, which are perhaps multiplied by some other moments. All of these other moments are less than 1 in absolute value. Therefore, we can write the following estimate:

$$\left| E \left((B_n A_n)^k \right) \right| \leq M'_k \max_{i \leq k} \left\{ a_i^{(n)} \right\},$$

where M'_k is the number of terms in the polynomial. If k is fixed and n is growing, then the assumption that A_n converges in distribution to the uniform law implies that $\max_{i \leq k} \left\{ a_i^{(n)} \right\}$ converges to zero. Therefore, all moments of $B_n A_n$ converge to zero as $n \rightarrow \infty$, and therefore, by Lemma 8 and Proposition 9, the sequence $B_n A_n$ converges in distribution to the uniform law. A similar argument proves that $A_n B_n$ converges in distribution to the uniform law. QED.

Lemma 11 *Suppose that B is a unitary operator, $\{A_n\}$ is a sequence of unitary operators, B is free from each of A_n , $E(B) \neq 0$, and the sequence A_n does not converge to uniform law. Then the sequence of products BA_n does not converge to the uniform law.*

Proof: The condition that the sequence A_n does not converge to the uniform law means that for some fixed k the sequence of k -th moments of A_n does not converge to zero as $n \rightarrow \infty$. Let k be the smallest of these indices. By selecting a subsequence we can assume that $|E(A_n^k)| > \alpha > 0$ for all n . Consider $E((BA_n)^k)$:

$$E((BA_n)^k) = [E(B)]^k E(A_n^k) + \dots,$$

The number of the terms captured by \dots is finite and depends only on k . Each of these terms includes at least one of $E(A_n^i)$ where $i < k$, and other multipliers in this term are less than 1 in absolute value. Therefore, each of these terms converges to zero. Hence, for any $\varepsilon > 0$, there exist such N that for all $n > N$, the sum of the terms captured by \dots is less than ε in absolute value. Take $\varepsilon = |E(B)|^k \alpha / 2$. Then for $n > N$, we have:

$$\left| E((BA_n)^k) \right| \geq |E(B)|^k \alpha / 2.$$

Therefore, the sequence of products BA_n does not converge to the uniform law. QED.

Lemma 12 *Suppose that B is a unitary random variable, $\{A_n\}$ is a sequence of unitary random variables, B is free from each of A_n , B is not uniform, and the sequence of expectations $E(A_n)$ does not converge to zero. Then the sequence of products BA_n does not converge to the uniform law.*

Proof: By selecting a subsequence we can assume that $|E(A_n)| > \alpha > 0$ for all n . The assumption that B is not uniform means that for some $k \geq 1$, $E(B^k) \neq 0$. Let k be the smallest of such k . Consider $E((BA_n)^k)$:

$$E((BA_n)^k) = [E(A_n)]^k E(B^k) + \dots,$$

Each of the terms in \dots includes one of $E(B^i)$ where $i < k$. Therefore, all terms in \dots are zero. Hence,

$$\left| E((BA_n)^k) \right| = \left| [E(A_n)]^k E(B^k) \right| > \alpha^k |E(B^k)|.$$

Therefore, the sequence of products BA_n does not converge to the uniform law. QED.

5 Analysis

We use the following notation: ψ_i and S_i denote ψ - and S -functions for variables X_i (and measures μ_i), and $\psi_{(n)}$ and $S_{(n)}$ denote these functions for variables Π_n (and measures $\mu^{(n)}$).

Case I: $a_{(n)} \not\rightarrow 0$.

Since $E(\Pi_n) = a_{(n)}$, therefore, if $a_{(n)} \not\rightarrow 0$, then $E(\Pi_n) \not\rightarrow 0$. Hence, Π_n cannot converge to the uniform measure on the unit circle.

Case II $a_{(n)} \rightarrow 0$, and there are at least two indices i and j such that $a_i = a_j = 0$.

Assume without loss of generality that $j > i$. Consider Π_n with $n \geq j$ and define $X =: X_1 \dots X_i$ and $Y =: X_{i+1} \dots X_n$. Then $\Pi_n = XY$, and $E(Y) = E(X) = 0$. Using Lemma 7, we obtain that $\left| E[(\Pi_n)^k] \right| = 0$ for every $k > 0$. Therefore, the ψ -function of Π_n is zero, and Π_n has the uniform distribution on the unit circle.

Case III $a_{(n)} \rightarrow 0$, and for all i , $a_i > 0$.

Subcase III.1 $\liminf a_i = 0$.

In this case we can find a subsequence a_{n_i} that monotonically converges to zero.

Now, consider Π_j , where $j \in [n_i, n_{i+1})$. Then we can write $\Pi_j = XY$, where $X = X_1 \dots X_{n_i-1}$, and $Y = X_{n_i} \dots X_j$. Then $EX \leq a_{n_i-1}$ and $EY \leq a_{n_i} \leq a_{n_{i+1}}$.

Applying Lemma 7 we get

$$\left| E(\Pi_j^k) \right| \leq M_k a_{n_{i+1}}.$$

This implies that for a fixed k , $\left| E(\Pi_j^k) \right|$ approaches zero as $j \rightarrow \infty$. By Lemma 8 and Proposition 9, this establishes that Π_j converges to the uniform law.

Case III $a_{(n)} \rightarrow 0$, and for all i , $a_i > 0$

Subcase III.2 $\liminf a_i = \underline{a} > 0$.

Let us choose such an a that $0 < a < \underline{a}$. Starting from some j_0 , $a_j \in (a, 1)$. Let $\tilde{\Pi}_n = X_{j_0} \dots X_{n+j_0-1}$. Then, by Lemmas 10 and 11, $\tilde{\Pi}_n$ converges to the uniform law if and only if Π_n converges to the uniform law. Hence, without loss of generality we can restrict our attention to the case when $a_k \in (a, 1)$ for all k .

Lemma 13 *Suppose $1 \geq a_k > 0$ for all k , and let $\alpha_i =: 1 - a_i$. Then $\prod_{i=1}^n a_i \rightarrow 0$ if and only if $\sum_{i=1}^n \alpha_i \rightarrow \infty$.*

This is a standard result. For a proof see Section 2.7 in Whittaker and Watson [1927].

Since $\log(1 - \alpha_i) \leq -\alpha_i$, we also have the following estimate, which we will find useful later.

$$\prod_{i=1}^n a_i \leq \exp\left(-\sum_{i=1}^n \alpha_i\right). \quad (4)$$

To prove convergence to the uniform law, we have to establish that for every $k > 0$ the coefficient $c_k^{(n)}$ in the Taylor expansion of function $\psi_{(n)}(z)$ approaches zero as $n \rightarrow \infty$. We know from Lemma 6 that

$$kc_k^{(n)} = \operatorname{res}_{z=0} \frac{1}{\left[\psi_{(n)}^{-1}(z)\right]^k};$$

therefore, our main task is to estimate this residual. This is the same as estimating the coefficient before the term z^{k-1} in the Taylor expansion of

$$f(z) = \left[\frac{z}{\psi_{(n)}^{-1}(z)}\right]^k.$$

We will approach this problem by using the Cauchy inequality (see Section 5.23 in Whittaker and Watson [1927]). Applied to the coefficient before z^{k-1} in the Taylor expansion of $f(z)$, this inequality says that

$$\left|kc_k^{(n)}\right| \leq \frac{M(r)}{r^{k-1}}, \quad (5)$$

where $r > 0$ is such that $f(z)$ is analytic inside $|z| = r$, and

$$M(r) =: \max_{|z|=r} |f(z)|.$$

It is easy to check that the constant in the Taylor expansion of $z/\psi_{(n)}^{-1}(z)$ is $a_{(n)}$. So $M(0) = a_{(n)}$, which approaches zero as $n \rightarrow \infty$. The main question is how large we can take r , so that $M(r)$ remains relatively small. In other words, we want to minimize the right-hand side of (5) by a suitable choice of r .

Proposition 14 Suppose that $EX_i = a_i > a$ for each i and that $a_{(n)} =: \prod_{i=1}^n a_i \rightarrow 0$. Let $\alpha_i = 1 - a_i$. Then for all sufficiently large n , the following inequality holds:

$$|c_k^{(n)}| \leq \left(\frac{C}{a^2}\right)^k \left[\left(\sum_{i=1}^n \alpha_i\right) \exp\left(-\sum_{i=1}^n \alpha_i\right) \right]^k,$$

where $C = 2^{17}$.

Proof: The main tool in the proof is the following proposition:

Proposition 15 Suppose that $\alpha_i =: 1 - a_i < 1 - a$ for each i , and that z and n are such that

$$|z| \leq \frac{a^2}{6684} \min \left\{ 1, \left(\sum_{i=1}^n \alpha_i\right)^{-1} \right\}.$$

Then,

$$\left| \frac{z}{\psi_{(n)}^{-1}(z)} \right|^k \leq (2e^2)^k \left(\prod_{i=1}^n a_i\right)^k.$$

We will prove this proposition in the next section and assume for now that it holds.

Let n_0 be so large that $\sum_{i=1}^{n_0} \alpha_i > 1$. (We can find such n_0 because by Lemma 13, $\sum_{i=1}^n \alpha_i \rightarrow \infty$ as $n \rightarrow \infty$.) In particular, this implies that $\sum_{i=1}^n \alpha_i > 1$ for every $n \geq n_0$. Define $r_n =: a^2 (\sum_{i=1}^n \alpha_i)^{-1} / 6684$. Then, using Proposition 15 and formulas (5) and (4), we get:

$$\begin{aligned} |kc_k^{(n)}| &\leq (2e^2)^k \left(\prod_{i=1}^n a_i\right)^k \left(\frac{6684}{a^2} \sum_{i=1}^n \alpha_i\right)^{k-1} \\ &\leq \left[\frac{2^{17}}{a^2} \left(\sum_{i=1}^n \alpha_i\right) \exp\left(-\sum_{i=1}^n \alpha_i\right) \right]^k, \end{aligned}$$

provided that $n \geq n_0$. QED.

Using Lemma 13, we get the following Corollary:

Corollary 16 If the assumptions of Proposition 14 hold, then for each k , the coefficient $c_k^{(n)} \rightarrow 0$ as $n \rightarrow \infty$.

This Corollary shows that in Case III.2 the product Π_n converges to the uniform law.

Case IV $a_{(n)} \rightarrow 0$, and there exists exactly one index i , such that $a_i = 0$.

First, we want to show that without loss of generality we can assume in this case that $a_1 = 0$, and $a_k > 0$ for all $k > 1$. Indeed, suppose $a_i = 0$ for $i > 1$,

and $a_j > 0$ for $j < i$. Let $X = X_1 \dots X_{i-1}$ and let $\tilde{\Pi}_n = X_i \dots X_{i+n-1}$. Then $E(X) \neq 0$, and using Lemmas 10 and 11, we conclude that Π_n converges to the uniform law if and only if $\tilde{\Pi}_n$ converges to the uniform law.

Subcase IV.1 X_1 has the uniform distribution.

In this case all moments of X_1 are zero, i.e., $E(X_1^k) = 0$ for all $k > 0$, and Proposition 4 implies that all moments of Π_n are zero. Therefore, Π_n is uniform for all n .

Subcase IV.2 X_1 does not have the uniform distribution, and $\prod_{k=2}^n a_n \rightarrow 0$ as $n \rightarrow \infty$.

By Case III, the product $X_2 \dots X_n$ converges to the uniform law, and using Lemma 10, we conclude that Π_n also converges to the uniform law.

Subcase IV.3 X_1 does not have the uniform distribution and $\prod_{k=2}^n a_n \not\rightarrow 0$ as $n \rightarrow \infty$.

Applying Lemma 12 to $B = X_1$ and $A = X_2 \dots X_n$, we conclude that Π_n does not converge to the uniform law.

6 Proof of Proposition 15

Let

$$f(z) =: \left(\frac{z}{\psi_{(n)}^{-1}(z)} \right)^k.$$

Using Theorem 5, we can write this function as follows:

$$f(z) = \left(\frac{z^n}{(1+z)^{n-1}} \prod_{i=1}^n \frac{1}{\psi_i^{-1}(z)} \right)^k. \quad (6)$$

We want to estimate $|f(z)|$ for all sufficiently small z . We start with some auxiliary estimates, which will later allow us to estimate $\psi_i(z)$, and then $\psi_i^{-1}(z)$ for small z .

Lemma 17 *Suppose μ is a probability measure on $[-\pi, \pi)$ such that*

$$\left| \int_{-\pi}^{\pi} (e^{i\theta} - 1) d\mu(\theta) \right| \leq \alpha. \quad (7)$$

Then, i)

$$\int_{-\pi}^{\pi} \theta^2 d\mu(\theta) \leq \frac{\pi^2}{2} \alpha < 5\alpha;$$

ii)

$$\left| \int_{-\pi}^{\pi} \theta d\mu(\theta) \right| \leq \left(1 + \frac{\pi^3}{12} \right) \alpha < 4\alpha, \text{ and}$$

iii) if $k > 2$, then

$$\int_{-\pi}^{\pi} |\theta|^k d\mu(\theta) \leq \frac{\pi^k}{2} \alpha.$$

Proof: Condition (7) implies that

$$\int_{-\pi}^{\pi} (1 - \cos(\theta)) d\mu(\theta) \leq \alpha$$

and that

$$\left| \int_{-\pi}^{\pi} \sin(\theta) d\mu(\theta) \right| \leq \alpha.$$

Since $1 - \cos \theta \geq (2/\pi^2) \theta^2$, from the first of these inequalities we infer that:

$$\int_{-\pi}^{\pi} \theta^2 d\mu(\theta) \leq (\pi^2/2) \alpha,$$

which proves claim i) of the lemma.

Next, note that $|\sin \theta - \theta| \leq |\theta|^3/6$, and that

$$\frac{1}{6} \int_{-\pi}^{\pi} |\theta|^3 d\mu(\theta) \leq \frac{\pi}{6} \int_{-\pi}^{\pi} \theta^2 d\mu(\theta) \leq \frac{\pi^3}{12} \alpha.$$

Therefore,

$$\begin{aligned} \left| \int_{-\pi}^{\pi} \theta d\mu(\theta) \right| &\leq \left| \int_{-\pi}^{\pi} \sin(\theta) d\mu(\theta) \right| + \left| \int_{-\pi}^{\pi} (\theta - \sin(\theta)) d\mu(\theta) \right| \\ &\leq \alpha + \frac{\pi^3}{12} \alpha. \end{aligned}$$

This proves claim ii) of the lemma.

For claim iii), note that

$$\int_{-\pi}^{\pi} |\theta|^k d\mu(\theta) \leq \pi^{k-2} \int_{-\pi}^{\pi} |\theta|^2 d\mu(\theta) \leq \frac{\pi^k}{2} \alpha.$$

QED.

Lemma 18 Suppose Condition (7) holds, and k is a positive integer. Then

$$\left| \int_{-\pi}^{\pi} (e^{ik\theta} - 1) d\mu(\theta) \right| \leq 7k^3 \alpha.$$

Proof: First, remark that $1 - \cos(k\theta) \leq (k\theta)^2/2$ and therefore

$$\begin{aligned} \left| \int_{-\pi}^{\pi} (\cos k\theta - 1) d\mu(\theta) \right| &\leq \frac{k^2}{2} \int_{-\pi}^{\pi} \theta^2 d\mu(\theta) \\ &\leq \frac{\pi^2 k^2}{4} \alpha. \end{aligned}$$

Next, we will use $|\sin(k\theta) - k\theta| \leq (k|\theta|)^3/6$ and write

$$\begin{aligned} \left| \int_{-\pi}^{\pi} \sin(k\theta) d\mu(\theta) \right| &\leq \left| \int_{-\pi}^{\pi} k\theta d\mu(\theta) \right| + \left| \frac{1}{6} \int_{-\pi}^{\pi} (k|\theta|)^3 d\mu(\theta) \right| \\ &\leq k \left(1 + \frac{\pi^3}{12} \right) \alpha + \frac{1}{6} k^3 \frac{\pi^3}{2} \alpha \\ &\leq k^3 \left(1 + \frac{\pi^3}{6} \right) \alpha. \end{aligned}$$

Consequently,

$$\begin{aligned} \left| \int_{-\pi}^{\pi} (e^{ik\theta} - 1) d\mu(\theta) \right| &\leq \alpha \sqrt{\frac{\pi^4 k^4}{16} + k^6 \left(1 + \frac{\pi^3}{6} \right)^2} \\ &\leq 7k^3 \alpha. \end{aligned}$$

QED.

Lemma 19 *Let X be unitary and $EX = a > 0$. If $|z| \leq 1/2$ and $1 - a \leq \alpha$, then*

$$\left| \psi_X(z) - \frac{az}{1-z} \right| \leq 716\alpha |z|^2.$$

Proof: We can write:

$$\psi_X(z) - \frac{az}{1-z} = \sum_{k=2}^{\infty} (E(X^k) - a) z^k.$$

Therefore, using Lemma 18, we estimate:

$$\begin{aligned} \left| \psi_X(z) - \frac{az}{1-z} \right| &\leq \sum_{k=2}^{\infty} (|E(X^k) - 1| + |1 - a|) |z|^k \\ &\leq 7\alpha |z|^2 \sum_{k=0}^{\infty} [(k+2)^3 + 1/7] |z|^k \\ &\leq 716\alpha |z|^2. \end{aligned}$$

(Note that 716 is the exact value of the sum $7 \sum_{k=0}^{\infty} [(k+2)^3 + 1/7] 2^{-k}$.) QED.

To derive a similar estimate for $\psi_X^{-1}(z)$, we need a couple of preliminary lemmas.

Lemma 20 *Suppose X is unitary and $EX = a > 0$. Then the function $\psi_X(z)$ has only one zero ($z = 0$) in the area $|z| < a/3$. If $|z| = a/3$, then $|\psi_X(z)| \geq a^2/6$.*

Proof: Write the following estimate:

$$\begin{aligned} |\psi_X(z) - az| &= \left| \sum_{k=2}^{\infty} E(X^k) z^k \right| \\ &\leq |z| \sum_{k=1}^{\infty} |z|^k = \frac{|z|}{1-|z|} |z| < \frac{a}{2} |z|, \end{aligned}$$

if $|z| < a/3$. By Rouché's theorem, $\psi_X(z)$ has only one zero in $|z| < a/3$. The second claim also follows immediately from this estimate. QED.

Lemma 21 *Suppose X is unitary and $EX = a > 0$. Then the function $\psi_X^{-1}(z)$ is analytical for $|z| < a^2/6$. If $|z| \leq a^2/12$, then*

$$|\psi_X^{-1}(z)| \leq \frac{2}{a} |z|.$$

Proof: Using Lagrange's formula, we can write

$$\psi_X^{-1}(z) = \frac{z}{a} + \sum_{k=2}^{\infty} c_k z^k,$$

where

$$c_k = \frac{1}{2\pi i} \frac{1}{k} \oint_{\gamma} \frac{du}{[\psi_X(u)]^k}.$$

By the previous lemma, we can use the circle with the center at 0 and radius $a/3$ as γ , and then we can estimate c_k as follows:

$$|c_k| \leq \frac{a/3}{k(a^2/6)^k} = \frac{2}{ka} \left(\frac{6}{a^2} \right)^{k-1}. \quad (8)$$

It follows that the power series for $\psi_X^{-1}(z)$ converges in $|z| < a^2/6$. If $|z| < a^2/12$, then we can estimate $\psi_X^{-1}(z)$:

$$\begin{aligned} |\psi_X^{-1}(z)| &\leq \frac{|z|}{a} \left(1 + a \sum_{k=2}^{\infty} |c_k| |z|^{k-1} \right) \\ &\leq \frac{|z|}{a} \left(1 + \sum_{k=2}^{\infty} \left(\frac{6}{a^2} \right)^{k-1} |z|^{k-1} \right) \\ &\leq \frac{|z|}{a} \frac{1}{1 - \frac{6}{a^2} |z|} \leq \frac{2}{a} |z|, \end{aligned}$$

where in the second line we used inequality (8). QED.

Lemma 22 *Let X be unitary and $EX = a > 0$. If $|z| \leq a^2/12$, and $\alpha \geq 1 - a$, then*

$$\left| \frac{\psi_X^{-1}(z)}{z/(a+z)} - 1 \right| \leq \frac{3342}{a^2} \alpha |z|.$$

Proof: First of all, by Lemma 21

$$|\psi_X^{-1}(z)| \leq \frac{2}{a} |z|$$

for $|z| \leq a^2/12$.

Now we use the functional equation for $\psi_X^{-1}(z)$:

$$\psi_X(\psi_X^{-1}(z)) = z.$$

If $|z| \leq a^2/12$, then $|\psi_X^{-1}(z)| \leq 2|z|/a \leq a/6 < 1/2$ and we can apply Lemma 19 to get:

$$\begin{aligned} \left| z - \frac{a\psi_X^{-1}(z)}{1 - \psi_X^{-1}(z)} \right| &\leq 716\alpha |\psi_X^{-1}(z)|^2 \\ &\leq 716\alpha \frac{4}{a^2} |z|^2 = \frac{2864\alpha}{a^2} |z|^2. \end{aligned}$$

Next, we write this as

$$\begin{aligned} |z - (a+z)\psi_X^{-1}(z)| &\leq |1 - \psi_X^{-1}(z)| \frac{2864\alpha}{a^2} |z|^2 \\ &\leq \frac{7}{6} \frac{2864\alpha}{a^2} |z|^2 < \frac{3342\alpha}{a^2} |z|^2. \end{aligned}$$

(In the second inequality we used the fact that $|\psi_X^{-1}(z)| \leq 1/6$ if $|z| \leq a^2/12$.)

It follows that

$$\left| \frac{\psi_X^{-1}(z)}{z/(a+z)} - 1 \right| \leq \frac{3342}{a^2} \alpha |z|.$$

QED.

Lemma 23 *Let $EX_i = a_i$ and assume that for each i , it is true that $a_i \geq a$. Assume also that $|z| \leq a^2/3342$ and let $\alpha_i = 1 - a_i$. Then*

$$\left| \prod_{i=1}^n \frac{1}{\psi_i^{-1}(z)} \right| \leq \frac{\prod_{i=1}^n a_i}{|z|^n} \prod_{i=1}^n \frac{1}{1 - c_i |z|} \left| \prod_{i=1}^n \left(1 + \frac{z}{a_i} \right) \right|,$$

where $c_i = 3342\alpha_i/a_i^2$.

Proof: From Lemma 22 we infer that

$$|\psi_i^{-1}(z)| \geq \left| \frac{z}{a_i} \frac{1}{1 + z/a_i} \right| \left(1 - \frac{3342\alpha_i}{a_i^2} |z| \right).$$

Multiplying these inequalities together and inverting both sides, we get the desired result. QED.

Lemma 24 *Under the assumptions of the previous lemma, the following inequality holds:*

$$|f(z)| \leq \left(|1+z| \prod_{i=1}^n a_i \prod_{i=1}^n \frac{1}{1-c_i|z|} \left| \prod_{i=1}^n \frac{1+z/a_i}{1+z} \right| \right)^k, \quad (9)$$

where $c_i = 3342\alpha_i/a_i^2$.

Proof: The claim of this lemma is a direct consequence of Lemma 23 and equality (6). QED.

We will estimate terms in the product on the right-hand side of (9) one by one.

Lemma 25 *Suppose that $\alpha_i =: 1 - a_i < 1 - a$ for each i , and that*

$$|z| \leq \frac{a^2}{6684} \min \left\{ 1, \left(\sum_{i=1}^n \alpha_i \right)^{-1} \right\}.$$

Then

$$\left| \prod_{i=1}^n \frac{1+z/a_i}{1+z} \right| \leq e.$$

Proof: We write:

$$\left| \prod_{i=1}^n \frac{1+z/a_i}{1+z} \right| = \exp \left(\operatorname{Re} \sum_{i=1}^n \log \left(1 + \frac{\alpha_i}{a_i} \frac{z}{1+z} \right) \right).$$

Recall that $\operatorname{Re} \log(1+u) \leq |u|$ if $|u| < 1$. Under our assumption about $|z|$, it is true that

$$\left| \frac{\alpha_i}{a_i} \frac{z}{1+z} \right| < 1.$$

Therefore we can write:

$$\begin{aligned} \left| \prod_{i=1}^n \frac{1+z/a_i}{1+z} \right| &\leq \exp \left(\left| \frac{z}{1+z} \right| \sum_{i=1}^n \frac{\alpha_i}{a_i} \right) \\ &\leq \exp \left(\frac{2}{a} |z| \sum_{i=1}^n \alpha_i \right) \\ &\leq e. \end{aligned}$$

QED.

Lemma 26 *Suppose that $\alpha_i =: 1 - a_i < 1 - a$ for each i , and that*

$$|z| \leq \frac{a^2}{6684} \min \left\{ 1, \left(\sum_{i=1}^n \alpha_i \right)^{-1} \right\}.$$

Then,

$$\prod_{i=1}^n \frac{1}{1 - c_i |z|} \leq e,$$

where $c_i = 3342\alpha_i/a_i^2$.

Proof: We use the inequality $\log(1 - u) \geq -2u$, which is valid for $u \in (0, 1/2)$, and write:

$$\begin{aligned} \prod_{i=1}^n \frac{1}{1 - c_i |z|} &= \exp \left(- \sum_{i=1}^n \log(1 - c_i |z|) \right) \\ &\leq \exp \left(2 |z| \sum_{i=1}^n c_i \right) \\ &\leq \exp \left[\frac{6684}{a^2} \left(\sum_{i=1}^n \alpha_i \right) |z| \right] \\ &\leq e. \end{aligned}$$

QED.

Finally, note that if $|z| \leq a^2/6684$, then $|1 + z| \leq 2$. Collecting all the pieces, we obtain that if

$$|z| \leq \frac{a^2}{6684} \min \left\{ 1, \left(\sum_{i=1}^n \alpha_i \right)^{-1} \right\}$$

then:

$$|f(z)| \leq (2e^2)^k \left(\prod_{i=1}^n a_i \right)^k.$$

This completes the proof of Proposition 15.

7 Conclusion

We have derived sufficient and necessary conditions for the product of free unitary operators to converge in distribution to the uniform law. If *essential convergence* denotes the situation when the partial products continue to converge even after an arbitrary finite number of terms are removed, then the necessary and sufficient condition for essential convergence is that the products $\prod_{i=k_0}^n EX_i$ converges to zero for all k_0 , that is, that the products of expectations essentially converge to zero. Essential convergence implies convergence. In addition, non-essential convergence can occur when there is either a term that has the uniform distribution, or there are two terms that have zero expectation. In the latter case convergence occurs because the product of these two terms has the uniform distribution.

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